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EXPERIMENTAL EVIDENCE OF NON-LINEARITY IN PLASTIC STRESS-STRAIN RELATIONS

Part I. Theoretical Considerations

Part II. Experimental Evidence

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Abstract of Part I

The most general plastic stress-strain relation for materials obeying Drucker's work-hardening criteria is specialized to commonly used forms. Particular attention is paid to the implications incurred by (a) the assumption of a loading function for these materials, and by (b) the assumption of the linearity of plastic strain increments in the stress increments.

Abstract of Part II

A detailed treatment of the experimental data obtained in the testing of two aluminum alloy thin-walled tubes is presented. Comparison is made of this analysis with the implications of two basic assumptions of incremental theories of plasticity for work-hardening materials. The conclusion in that the stress-strain relation for one of the tubes was linear in the increments of stress and plastic strain while that for the other was decidedly non-linear.

PART I

Introduction

Quite recently the mathematical theory of stress-strain laws of plasticity has attained a form of very broad generality. The theory embraces most previous theories as special cases. It is the purpose here to present this theory in its broadest aspect and to specialize it to the principal particular forms.

The theory is concerned with elastic-plastic materials that are independent of time and temperature effects. The material is assumed to be work-hardening in the sense described by Drucker (1). Most metals used in engineering practice at normal temperatures and rates of loading exhibit—at least to within the accuracy of normal experimentation—this independence of time and temperature. The statement of work-hardening referred to is a definition of a class of materials, and provides a mathematical description sufficiently general to include the phenomenon of work-hardening of metals as usually interpreted on the one hand and to give a logical extension to a completely general stressing of a body on the other.

Work-Hardening

As usually thought of, work-hardening means that given a material in a prescribed state of stress and strain, the rate of increase of both total and plastic work per unit volume per given change in any component of strain must be positive as the strain increments are increased in ratio (2). The concept has been generalized by Drucker (1) in the following manner:

Consider a material under a system of stresses σ^{A}_{ij} due to an agency A. Consider a system of stresses $\Delta\sigma^{B}_{ij}$ that are slowly applied and then removed from the body by an external agency B. "Slowly" is to be interpreted to imply that inertia, vibrational, and viscosity effects are not incurred. Work-hardening then requires that

(a) for all such added sets of stresses the material will remain in equilibrium, and that

(b) positive work be done by the external agency B during the application of the $\Delta\sigma^{B}_{ij}$

(c) the work done by the external agency B over the entire cycle be positive if plastic deformations have occurred. This work will be zero if and only if the strains are purely elastic.

The definition of work-hardening leads to the mathematical expressions

$\Delta\sigma^{B}_{ij}\ \Delta\varepsilon_{ij} > 0$ during the applications of the stresses; [1]

$\Delta\sigma^{B}_{ij}\ \Delta\varepsilon^{p}_{ij} \geq 0$ for the entire cycle, and [2]

$\Delta\sigma^{B}_{ij}\ \Delta\varepsilon^{p}_{ij} = 0$ if and only if $\Delta\varepsilon^{p}_{ij} = 0$,

provided, of course, $\Delta\varepsilon_{ij}$ can be separated from $\Delta\varepsilon^{p}_{ij}$ and $\Delta\varepsilon^{total}_{ij} =$

$\Delta\varepsilon^{p}_{ij} + \Delta\varepsilon^{e}_{ij}$. These expressions require stability in the strictest sense; no energy — even of infinitesimal order — may be obtained from the material by the agency B. If any plastic deformation is to take place, energy must be applied to the material by agency B. Furthermore the material will be in equilibrium for all systems $\Delta\sigma^{B}_{ij} + \sigma^{A}_{ij}$ that can be attained from σ^{A}_{ij}.

Fig. 1 illustrates these ideas by an example quite analagous to that used by Drucker. P represents a body assumed to be in the incipiently plastic state under loads kept constant during the following. The barbell at the right is pivoted at C so as to be free to turn about an axis perpendicular to the plane of the figure. A spring of zero mass is fastened to the top mass to symbolize the absence of impact of the upper weight against the block A when the barbell is released. The upper weight of the barbell is assumed very slightly larger than the lower so that in the position shown the barbell is in unstable equilibrium; pivot C and the spring are considered frictionless. If the barbell is now given an infinitesimal counterclockwise displacement, it will continue to turn slowly until the spring comes in contact with P. Inertia will compress the spring which will bring the system to rest and then cause the barbell to turn clockwise. However, the work-hardening definition above insures that the barbell will never go through its unstable equilibrium position, and that it will return to that position only if no plastic deformation occurred.

It is emphasized that the criteria (b) and (c) make no explicit assumptions with regard to stress-strain relations. They also do not hypothesize that a certain set of stress increments do or do not cause plastic deformation or loading. They merely state that if any additional set of stresses are slowly added to the exisiting stress state and slowly removed, then the resulting strains must satisfy the conditions enunciated.

Most General Stress-Strain Law

The most general stress-strain law that can be written then, under the condition that the material be independent of time and temperature effects, would be a functional relation between the components of σ_{ij}, ε_{ij} (both the elastic ε^{e}_{ij} and the plastic ε^{p}_{ij}) and the entire history of stress and strain. The only restrictions that would be imposed on the relation would be those implied in (b) and (c).

A pictorial representation that is frequently used may make the idea of the restriction clearer. Consider the nine components of σ_{ij} as the components of a cartesian vector. Consider the components of ε_{ij} as components of a cartesian vector referred to the same axes as the vector σ_{ij}. Then σ_{ij}, ε^{total}_{ij}, ε^{e}_{ij}, ε^{p}_{ij} or their increments or time rates of any of these may be considered vectors in this space. The only restriction on this general stress-strain law would be that for any stress-strain state of the material, $\Delta\sigma_{ij}$ and $\Delta\varepsilon_{ij}$ must make an acute angle with each other, and that in any loading cycle $\Delta\sigma_{ij}$

and $\Delta\varepsilon_{ij}^{p}$ must also make an acute angle with each other for non-zero $\Delta\varepsilon_{ij}^{p}$.

Such a law, however, would be totally unusable from an engineering point of view. Since each material differs from each other material, the only determination of the stress-strain law of a particular material would be an experimental one.

Loading Functions

Two additional almost universally adopted assumptions are employed in an attempt to make the problem more tractable. The first is that added as a proviso to the mathematical statement of work-hardening conditions (b) and (c): the plastic and elastic strains and associated stresses are distinguishable and independent, and their effects are linearly additive. The second is the hypothesis of the existence of a loading function. For each state of strain and history there exists a function $f(\sigma_{ij})$ and a number k such that plastic strains will ensue only upon reaching a state of stress $\bar{\sigma}_{ij}$ for which $f(\bar{\sigma}_{ij}) > k$. Here f is usually considered as a function of the stresses only in which the states of strain and history appear as parameters. The number k may also be dependent on the plastic strain and the plastic strain history.

$f(\sigma_{ij})$ may conveniently be thought of as a means of classifying points in stress space into three classes. P, E, and B. P is the set of all σ_{ij} for which $f(\sigma_{ij}) > k$. B is the set of σ_{ij} which are boundary points for P and the complement of P — i.e. points for which $f(\sigma_{ij}) \leq k$. Finally E is the set of points which is the complement of the set P+B. The points of E are said to be "inside" the boundary B while the set P is "outside".

A complete discussion of the implications of the hypothesis of a loading function is lacking at the present time. However, certain broad categories of functions have been used and quite general results have been obtained from them. The most inclusive yet presented is that of continuous functions of σ_{ij}.

For these functions the boundary set B is also continuous and forms a surface in σ_{ij} space -- commonly called the yield or loading surface. For all loading functions the set E is composed entirely of elastic states possible for the given state of stress and strain history. The assumption of continuity assures that the boundary points also belong to the set with which only elastic strains are associated. For this case loading is incurred for outward pointing stress increment (rate) vectors starting from any stress state representable by a point on the loading surface; unloading, for any inward pointing increment (rate) vector; and neutral loading for any increment vector tangent to the loading surface (3).

Drucker proved (4) that for continuous loading functions the work-hardening criteria implied that the set E + B is convex. Since the loading function is still a function of plastic strain and plastic strain history, the surface is, of course, still free to expand, translate, change shape, or perform any combination of these in stress space as loading proceeds. It is interesting to note that in the movement of the surface, the stress free point, $\sigma_{ij}=0$, may at some stage become an outside point.

Since the boundary is continuous,* it must consist only of smooth points (i.e. points at which there is a continuously turning tangent plane to the loading surface) and of points (each of which will be referred to hereafter as a "pointed vertex") which are on either corners or points of the loading surface. Convexity of the loading surface implies that the strain increment vector must be parallel to the outward normal at each smooth point of the loading surface, and that at each pointed vertex the strain increment vector must not make an obtuse angle with the stress increment vector that caused it.

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*In the proof of convexity the following reasonable though lengthy-to-state theorem was tacitly assumed:

Given: 1) a, any point of E
2) b, any point of B that can be reached by a path, γ, from a to b such that γ lies entirely in E (except, of course, for b)
3) $d\gamma$ (= c - b) any infinitesimal incremental stress vector that constitutes loading from b to c
4) Let f(c) determine a new boundary B' with inside E'.

Conclusion: There exists a path γ' contained in B' + E' joining c and a.

Physically speaking, this theorem merely states that if from a stress state σ_{ij} an external agency applies a set of stresses that involve only elastic strains and an infinitesimal plastic strain, then there exists a way to return to stress state σ_{ij} by means of stresses that involve only elastic changes of strain.

If this theorem is accepted, it is not strictly necessary to assume the continuity of the boundary since the proof of convexity given by Drucker now carries through, and since the boundary of a convex set is continuous.

Almost all loading functions of the isotropic and of the anisotropic type that have been proposed to date are included in the type for which Drucker proved convexity. Further assumptions concerning the particular materials for which the loading function is to be used influence the particular form of the loading function. Four broad categories have been used extensively:

First note that convexity was established for those materials for which time rate (i.e. viscosity) effects are absent. This assumption implies that if time rate terms appear in the stress-strain relation, the function displaying them must be homogeneous and of zero order in them.

Isotropic materials in the extended sense are those in which there are no directional properties in stress free material. For such materials stresses appear in the loading function only in forms expressible as functions of the mean normal pressure J_1 and of the invariants J_2 and J_3 of the stress deviation s_{ij}: $J_1 = 1/3\ \sigma_{ii}$, $J_2 = 1/2\ s_{ij}\ s_{ji}$, and $J_3 = 1/3\ s_{ij}\ s_{jk}\ s_{ki}$ where $s_{ij} = \sigma_{ij} - 1/3\ \sigma_{kk}\ \delta_{ij}$. Similarly the plastic strain should appear only in forms expressible as functions of the invariants I_1, I_2, and I_3 of ε^p_{ij} : $I_1 = \varepsilon^p_{ii}$, $I_2 = 1/2\ \varepsilon^p_{ij}\ \varepsilon^p_{ji}$, and $I_3 = 1/3\ \varepsilon^p_{ij}\ \varepsilon^p_{jk}\ \varepsilon^p_{ki}$. Finally the strain rates may appear in f only as functions K_1, K_2, and K_3 of the plastic strain rate $\dot{\varepsilon}^p_{ij}$: $K_1 = \dot{\varepsilon}^p_{ij}\ \dot{\varepsilon}^p_{ij}$, etc.

If the plastic deformation of a material is assumed to be incompressible, then by definition $I_1 = 0$. It follows also that $K_1 = 0$. The loading function must be independent of the mean normal pressure, all plastic stress-strain relations may be expressed as relations between plastic strain increments and stress deviations.

If an initially isotropic material is to display subsequent anisotropy in the unloaded condition, plastic strains must be included in the loading function (5,6).

Finally there is a large number of loading functions that are functions of stress alone, i.e. independent of the plastic strain. This is possibly the most popular group of all.

Two points are emphasized with regard to the above categories. The first is that it is not intentionally implied that these categories are exhaustive nor that they are mutually exclusive, but rather that most existing loading functions can be classified in one or more of them. The second is that the inferences do not apply necessarily to the value k which indicates the value the loading function must attain before plastic deformations occur.

Plastic Stress-Strain Relations

The existence of smooth "convex" loading functions effects a great simplification in the form of the stress-strain law. Since at smooth points a unique outward normal exists and since the plastic strain increment must be parallel to it, it follows that the law must be

$$d\epsilon_{ij}^{p} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad \text{for } f > k \text{ and } \dot{f} > 0 \qquad [3a]$$

and

$$d\epsilon_{ij}^{p} = 0 \qquad \text{otherwise.} \qquad [3b]$$

Here λ is a scalar multiplier that may depend on the stress, plastic strain and plastic stress-strain history. Since [3a] holds only for $\dot{f} > 0$ and since $\frac{\partial f}{\partial \sigma_{ij}}$ exists, [3] may be written

$$\begin{aligned} d\varepsilon^p_{ij} &= G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{pq}} d\sigma_{pq} \quad \text{for } f > k, \; \frac{\partial f}{\partial \sigma_{pq}} d\sigma_{pq} > 0 \\ d\varepsilon^p_{ij} &= 0 \qquad\qquad\qquad\quad \text{for } \frac{\partial f}{\partial \sigma_{pq}} d\sigma_{pq} \leq 0 \end{aligned} \qquad [4]$$

where G. is a scalar function of stress, etc. This is the most general form of the plastic stress-strain law compatible with Drucker's definition of work-hardening and the assumption of the existence of a continuous loading function (see footnote on page 9).

It should be noted that $\frac{\partial f}{\partial \sigma_{pq}} d\sigma_{pq}$ is not a complete differential of f except for those f's which are functions only of stress. Even in this eventuality the right hand side of [4] does not represent a complete differential except for certain paths of loading. For these paths of loading, the flow theories and deformation theories coincide. Many interesting cases of this coincidence have been studied in (5).

The particular form [4] is misleading in that superficially it appears that the plastic strain increments are linear functions of the stress increments. As pointed out previously, it is permissible for the increments to appear in the plastic stress-strain relation (i.e. in the coefficients of $d\sigma_{ij}$ for the form discussed here) provided they appear only in homogeneous functions of zero order. The assumption of linearity requires that the coefficients G and $\frac{\partial f}{\partial \sigma_{ij}}$ be independent of stress or of strain increments entirely.

The introduction of the linearity of the incremental plastic strain and stress increments into incremental plastic stress-strain laws must be regarded as an assumption insofar as proofs existing to date are concerned. This fact was either overlooked entirely or glossed over in a great deal of the development of plastic stress-strain relations (7). Furthermore, the assumption of linearity is entirely independent of the assumption of the existence of a loading function. It merely states that in the most general form the stress-strain relations may be written $d\epsilon_{ij} = A_{ijkl}\, d\sigma_{kl}$ where the A_{ijkl} are independent of the increments $d\sigma_{ij}$ or $d\epsilon_{ij}$ and that the effect of two different differential loadings $d\sigma_{ij}^{(1)}$ and $d\sigma_{ij}^{(2)}$ is the same as the combined effect of both loadings; i.e.

$$d\epsilon_{ij}^{p}\,(d\sigma_{pq}^{(1)}+d\sigma_{pq}^{(2)}) = d\epsilon_{ij}^{p}\,(d\sigma_{pq}^{(1)})+d\epsilon_{ij}^{p}(d\sigma_{pq}^{(2)}) \qquad [5]$$

It is interesting to contrast the general implications of the assumptions of the existence of a loading function and of the validity of linearity. The existence of a loading function puts restrictions on the possible directions for plastic strain increment vectors corresponding to a given stress increment vector at a point. The validity of linearity, on the other hand, puts restrictions on the magnitudes of the plastic strain increment vectors resulting from loading at a point.

The most general form of the stress-strain law that embraces both the existence of a smooth loading function and the validity of linearity is the same as [4] except that neither f nor G may be dependent on the increments of either the plastic strain or the stress.

If, however, a loading function has a pointed vertex, linearity at that point is impossible. This fact can be illustrated most simply in the case of plane stress referred to principal axes fixed in space, although the results can be generalized readily. For this case the stress state can be represented by a point in a two-dimensional drawing (Fig. 2). Here the loading surface becomes a curve. At a pointed vertex in the loading curve, the curve will be represented by the arcs γ_1 and γ_2 which meet in the pointed vertex P. The convexity assures that in some neighborhood of P the arcs γ_1 and γ_2 are smooth. It also assures that the "corner" will point outward. This implies that $\varphi > \pi$ where φ is the angle between the tangents T_1P and T_2P to γ_1 and γ_2 respectively. Let BB' be the bisector of φ and let A_1PA_2 be perpendicular to BPB' at P. Note that any vector $d\sigma_{ij}^{(1)}$ that lies between PA_1 and PT_1 constitutes loading. Any vector $d\sigma_{ij}^{(2)}$ lying between PA_2 and PT_2 also constitutes loading. Now it is clear that equation [5] cannot hold generally since it is possible to find $d\sigma_{ij}^{(1)}$ and $d\sigma_{ij}^{(2)}$ that individually constitute loading but together do not; e.g. if $d\sigma_{ij}^{(1)}$ is symmetric to $d\sigma_{ij}^{(2)}$ with respect to B'PB and both lie below A_1PA_2.

Work-hardening criteria [1] and [2] yield information on the location of the plastic strain increment vector. Equation [1] implies that the plastic strain increment vector shall not make an obtuse angle with any possible "elastic" stress vector whose end-point is the point from which loading takes place. Therefore, the plastic strain increment vector must be contained inside or on the surface formed by the normals to the smooth loading surface points in the neighborhood of the corners. Equation [2] implies that the plastic strain increment vector does not make an obtuse angle with the stress increment vector. This fact implies in turn that one direction of $d\varepsilon_{ij}^{p}$ cannot suffice for all loading directions from a pointed vertex (Fig.3).

Although loading functions with pointed vertices apparently lead to mechanical difficulties of manipulation, it should not be concluded that this is always so or that the smooth loading functions are preferred. Indeed, with the notable exception of v. Mises loading function J = k (8) almost all attempts to fit experimental data with smooth loading functions lead to expressions involving J_3 (9), and these expressions are almost universally cumbersome for any except the simplest loading paths. Tresca's maximum shear criterion on the other hand has enjoyed a popularity comparable to that of $J_2 = k$ because of its ease of application (10); yet this function has corners on its loading surface. Of course in its completely general analytic form J_3 appears here also; however, as commonly employed, it is possible to use it without reference to J_3. It is also noteworthy that the recent development of a mathematical theory of plasticity based on slipping in individual grains permits and in fact requires pointed vertices (11). Furthermore, the recent development at Brown of plastic stress-strain relations for soil mechanics also permit corners.

Work-Hardening and Stress-Strain Laws

From a mathematical standpoint the idea of an "increment" is not always clear. For this reason it is often desirable to define loading and allied concepts in terms of time rates. When a material has applied to it stress rates that give rise to plastic deformations, those stress rates are said to constitute "loading". If a material exists in such a state that loading is possible from that state (incipiently plastic state) stress rates that lead to states from which loading is impossible are said to constitute "unloading". If a material is in the incipiently plastic state, non-zero stress rates that constitute neither loading nor unloading are termed "neutral loading".

Prager has stated four criteria that a useful mathematical stress-strain relation for plastic materials should satisfy (12): irreversibility, continuity, consistency and uniqueness. Irreversibility requires that the work done by the stresses on the plastic strains be positive. Continuity requires that any neutral loading may be considered a limiting case of either loading or unloading. Consistency requires that any loading from a given stress state leads to stress states from which loading is again possible. Finally uniqueness requires a unique determination of stress rates throughout a body provided the mechanical state (including stress-strain history) of a body and the system of surface traction rates on the body be given. The question arises, "Does the assumption of work-hardening in any way guarantee these conditions?"

Irreversibility is assured immediately by work-hardening condition (c) that states that the total work done in any cycle must be greater than or equal to zero, the equality holding only if no plastic deformation has occurred during the cycle.

It is not apparent at this time that the assumption of work-hardening implies uniqueness of the above stated boundary value problem. Certainly it cannot unless a stress-strain law gives uniqueness in the small. Even then it is not apparent that the boundary value problem is uniquely satisfied. The assumption of linearity gives uniqueness provided the material is loaded throughout since it can be shown by methods analogous to those used to prove the theorem of virtual work that if two sets of stress rates and strain rates satisfy equilibrium and compatibility and the boundary value problem given above, then

$$\int_V (\dot{\sigma}_{ij}^{(1)} - \dot{\sigma}_{ij}^{(2)}) (\dot{\varepsilon}_{ij}^{(1)} - \dot{\varepsilon}_{ij}^{(2)})\, dv = 0$$

But since both $\dot{\sigma}_{ij}^{(1)}$ and $\dot{\sigma}_{ij}^{(2)}$ constitute loading, work-hardening and linearity imply that the integrand is always positive. Hence the two solutions must coincide. It is not obvious that the result is still valid if, say, $\sigma_{ij}^{(1)}$ constitutes loading and $\sigma_{ij}^{(2)}$ unloading. If, however, a loading function is also assumed, linearity will guarantee that it implies a smooth surface in stress space. Under these conditions the proof offered by

Hodge and Prager (13) now holds, and uniqueness follows even though f is not necessarily a function of J_2 and J_3 only.

Finally continuity and consistency are properties of the stress-strain law itself and do not depend on the assumption of work-hardening.

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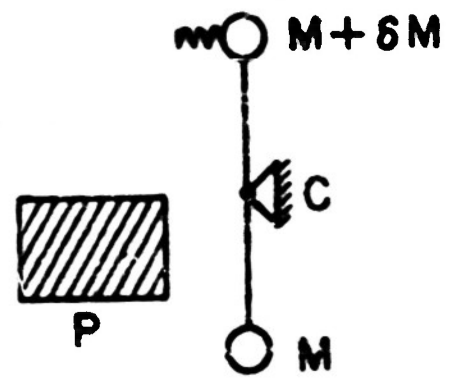
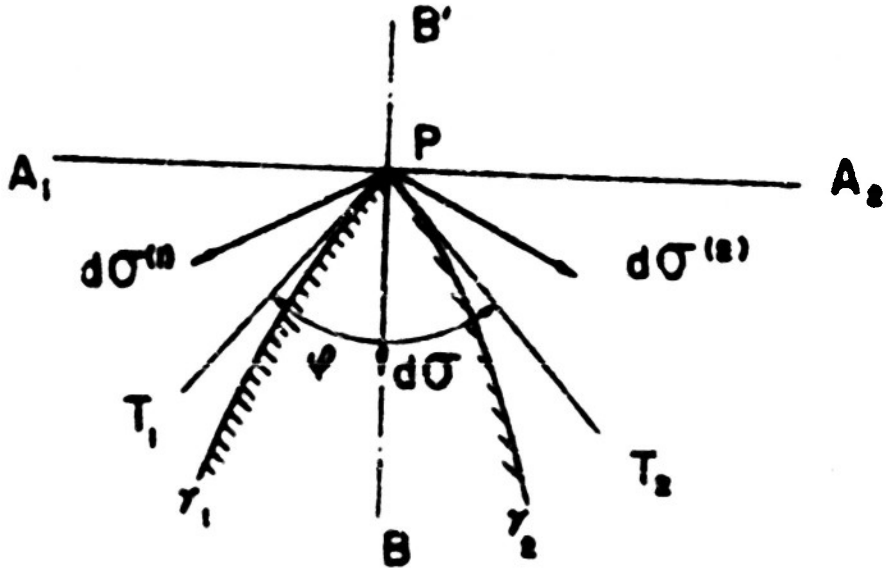


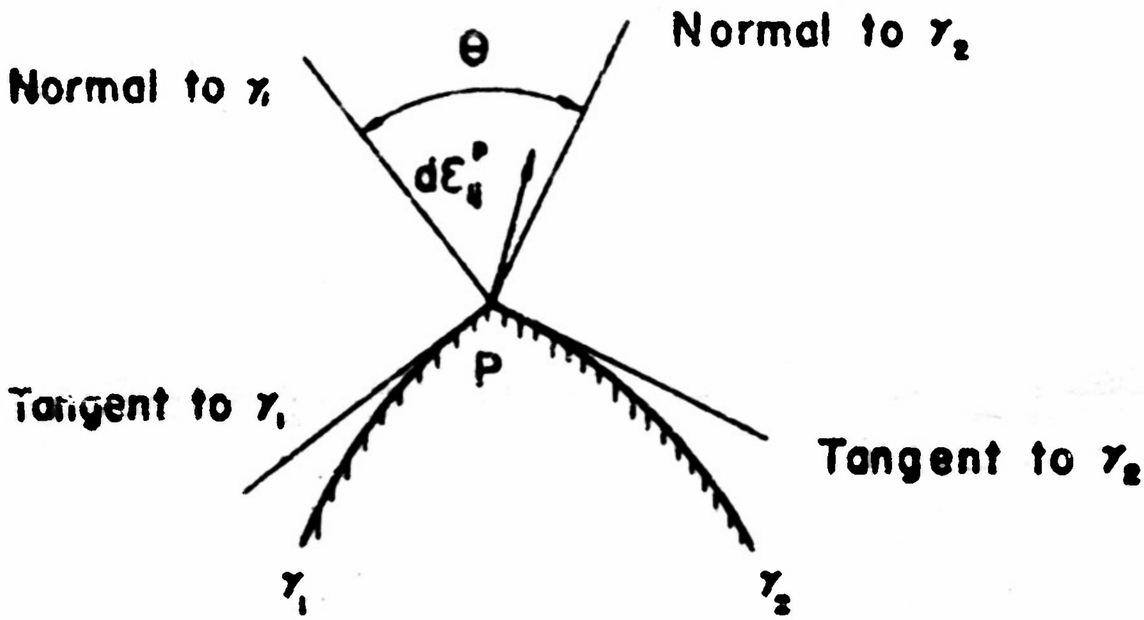
Dumbbell cannot bounce past neutral equilibrium pcint. It cannot even attain it if dumbbell causes plastic deformation.

FIG. 1



$d\sigma^{(1)}$ and $d\sigma^{(2)}$ constitute loading. $d\sigma = d\sigma^{(1)} + d\sigma^{(2)}$ does not. Linearity is impossible.

FIG. 2



$d\varepsilon_{ij}^{p}$ must lie in region θ

FIG. 3

PART II

Introduction

Most efforts to describe mathematically the stress-strain relation of a work-hardening material in which time and temperature effects are absent have adopted the hypothesis of the existence of a loading function. A loading function is a function of the state of stress, strain and history that determines when and if additional plastic strains will take place. In stress space any stress state is representable by a joint, or, alternatively, by a vector from the origin to the point (Fig. 1). Similarly, increments of stress from a given stress state are representable by incremental vectors from the existing stress point. Let f be a loading function. In stress space f = c is a surface called the yield or loading surface. Drucker has shown that his work-hardening criteria imply that the loading surface is convex (1). The interior or inside of the loading surface is that portion that originally contained the zero-stress point.

Let σ be a stress state, and let dσ be a stress increment. If σ lies on the loading surface, and if dσ points toward the outside of f = c, then plastic deformation will occur. Such increments of stress constitute loading. If dσ lies on the loading surface, then dσ constitutes neutral loading (2). Finally, if dσ points toward the inside, dσ constitutes unloading. In the last two cases all deformations are elastic. In the first case, of course, elastic deformation occurs as well as plastic. All stress increments lying within or on the loading surface are accompanied by purely elastic changes.

For ideal plasticity the loading function is fixed in stress space. When the stress point reaches the yield surface, uncontained plastic deformation can take place; but this deformation has no effect on the yield surface. For a work-hardening material, however, the loading surface moves locally with the stress point during loading. The movement of the surface may be an expansion, translation, change of shape, or any combination of these.

Let the strain space corresponding to a given stress space be superimposed on the latter in such a manner that the corresponding axes coincide. Let the stress point be on the loading surface, and let the increment of stress constitute loading. Prager showed that if the loading surface had a continuously turning tangent plane at the stress point (i.e. was "smooth"), then the increment of plastic strain would be parallel to the outward normal to the loading surface at the stress point (3). Stated again, the direction of the plastic strain increment vector is determined by the direction of the normal to the loading surface if the normal is unique.

To fix the ideas under discussion as well as to prepare for the arguments to follow, consider a plane stress state referred to its principal axes. Such a stress state may be completely described by a point in a two-dimensional representation of the stress space. The loading surface will be represented by a curve - the curve actually being the intersection of the loading surface and the plane determined by the two non-zero principal stresses.

Consider a stress point on the loading curve(surface). If the loading curve is smooth at the point (Fig.2a) there will be a unique outward unit normal, n. Any incremental plastic strain vector arising from loading from the stress point will be parallel to n. Loading, neutral loading, or unloading will be determined by whether the stress increment vector makes respectively an acute, right, or obtuse angle with the normal.

Contrast this situation with that in which the stress point is at a pointed vertex of the loading curve (Fig.2b). At such a point "the normal to the loading surface" is not defined. Furthermore, the direction of the plastic strain increment vector is likewise undefined. It can be shown by the use of Drucker's work-hardening criteria (4) that (a) there is more than one possible direction, (b) one direction is not sufficient, and (c) the plastic straining direction must lie between or on the normals to the intersecting loading surfaces that form the pointed vertex (i.e. in Fig. 1b, on m which is perpendicular to S, or on n which is perpendicular to T, or in the region between them).

It is to be noted that the determination of the strain increment vector is not complete, however, until its magnitude as well as its direction is known. The magnitude is fixed mathematically by some hypothesis relating the strain increments and the stress increments. The most common one is that the strain increments are linear functions of the stress increments.

A previous paper (5) reported briefly the results of an experimental program investigating the stress-strain relations of a thin-walled tube. The primary purpose of the paper was to present the experimental techniques and details. Insofar as the analysis was carried out at that time, it was felt that the conclusion was justified that the direction of the strain increments were independent of the direction of the stress increments. This conclusion is another way of stating the existence of a smooth loading function. It was pointed out there that the question of linearity would take additional analysis.

The purpose of this paper is to present a much more sensitive analysis which confirms the conclusion stated above in the case of one of the tubes but not the other. As a consequence, linearity is not always attained.

Linearity

Linearity applied to the plastic incremental stress-strain relations refers only to the increments of the stresses and plastic strains. It means that if two stress increments $d\sigma_1$ and $d\sigma_2$ both constitute loading from a stress state, $d\epsilon_1$ and $d\epsilon_2$ being the associated plastic strain increments, then the increment $d\sigma_1$ plus $d\sigma_2$ constitutes loading and the resulting plastic strain is $d\epsilon_1$ plus $d\epsilon_2$. This relation implies that for $d\sigma_{ij}$ that constitutes loading

$$d\epsilon_{ij}^{p} = A_{ijkl}\, d\sigma_{kl} \qquad [1]$$

in which the A_{ijkl} are explicitly assumed to be independent of the increments of stress and strain, but are otherwise unrestricted insofar as the assumption of linearity is concerned. In [1] ϵ_{ij}^p is the plastic strain tensor and σ_{ij} is the stress tensor. In plane stress referred to principal axes fixed in space it is sufficient to consider two members of equation [1]:

$$d\epsilon_y^p = A_x d\sigma_2 + B_x d\sigma_y \qquad [2]$$
$$d\epsilon_y^p = A_y d\sigma_2 + B_y d\sigma_y$$

where x and y are fixed directions. Since the experimental work to follow was analyzed under the assumption of plane stress, attention is restricted to equations [2], although all comments made concerning [2] can easily be generalized in order to apply to [1].

It should be emphasized that linearity and the existence of a loading function are both assumptions; either may be made without the other. However, if, for instance, linearity has been accepted, the addition of the assumption of a smooth loading surface implies a relation between the A's and B's since the direction of the strain increment is determined by the loading surface. The relation is

$$A_x/A_y = B_x/B_y \qquad [3]$$

Again, since at smooth points of the loading surface the direction of the strain increment is orthogonal to the surface:

$$B_x = A_y \qquad [4]$$

the relations [3] and [4] do not, of course, alter the fact that the magnitude of the strain-increment vector is determined by a linear combination of stress increments. On the other hand the addition of the assumption of linearity everywhere to that of the existence of a loading function imposes conditions on the allowable forms of the loading surfaces: they must be smooth everywhere. This last fact follows from the possibility at a corner of picking two increments of stress each constituting loading, but whose sum does not constitute loading. Hence the sum of the corresponding individual plastic strains would not equal the zero plastic strain corresponding to the sum of the stress increments. Thus the condition of linearity would be violated.

In any case the hypothesis of both the existence of a loading function and the validity of linearity implies equations [3] and [4] and the smoothness of the loading surface. It further implies that the magnitude of the strain increment vector is given by a linear combination of the stress increments. These implications afford a method of investigating non-linearity. If the loading surface is not smooth or if either of equations [3] or [4] is not satisfied, then the loading surface does not exist or the relation is not linear. If the existence of a loading surface is accepted the conclusion would be non-linearity. Again if the magnitudes of the strain increments are not linear functions of the stress increments, the relation cannot be linear - this result, of course, is independent of the existence or non-existence of a smooth loading function.

<u>Description of Experiments</u>

The experiments consisted in subjecting thin-walled tubes to simultaneous and independently varying internal pressure and longitudinal pull (referred to also as "load"). The test schedule called for loading by longitudinal pull into the plastic region, the pressure remaining meanwhile at a constant small reference value. After this state had been attained, both the load and the pressure were varied independently.

It is known that v. Mises' yield criterion ($J_2 = k$) is a good approximation to the actual loading function. By this criterion for the load and pressure ranges used here the increment in the loading function caused by an increase of 10 lb. in the load is two hundred times as large as the corresponding increment caused by a 10 psi increase in the pressure. Insofar as was possible, therefore, pains were taken to insure that the pull was always increased during the test to insure that the material was always loaded. The actual form of the loading path is given in Fig. 3b and 4b.

The tubes were machined with a 9-inch straight cylindrical section the central five inches of which constituted the test gage length. The outside diameter of the central section was 2.200 inches and the wall thickness was 0.100 inches.

During a test run the following data were recorded: the longitudinal pull on the tube, the pressure within the tube, the change in length of the 5-inch gage length, and the change of diameter of the tube at each end of the 5-inch gage length.

The tube dimensions were held within 0.0005 inches both in diameter and in wall thickness over the entire 5-inch gage length. The loads were measured to within $\pm$ 20 lb. The pressure was recorded within $\pm$ 5 psi. The longitudinal change in length was magnified 19.8 times. The change in diameter of the tube was magnified 10.9 times. The change in diameter used in the test analysis here was the average of the two readings taken at the ends of the gage length. Both longitudinal and diametral changes were read after magnification on dial gages with a least increment of 0.0001 inch.

For complete detail and description of the experimental procedure see (3).

Analysis of the Data

Although not explicitly stated, the discussion of stress-strain relations so far has been for a point of a material body. In a physical test it is clearly necessary to consider the behavior of a region of a body. Measuring quantities over a finite distance immediately raises the question of whether or not the aggregate properly reflects the behavior of the single points. The attitude adopted here was that the best approximation that could be obtained for the behavior of an arbitrary particle was the average behavior of a large group of particles subjected as nearly as possible to the same conditions. Inherent in this attitude is the acceptance of a statistical homogeneity in the material. Although such homogeneity was not completely realized in the tubes used here, it was felt that the deviations from it were small enough not to affect the conclusions. It should be pointed out that in the case of metals non-homogeneity effects are minimized by the use of large gage lengths.

No mention has been made of isotropy because its presence or absence does not affect the arguments if the body is reasonably homogeneous.

It should be remarked at the outset that the pressures involved are small compared to the loads. The variations of both the pressure and the load are small compared to the existing load state. Hence, pressure and load variations were considered to be infinitesimal in the analysis. Thus it was assumed that if [illegible], were valid, equations [3] and [4] would not be violated within the sensitivity of the experiment for any pressure loop (Fig. 3b or 4b).

The purpose of the tests was to check the validity of the assumption of linearity, which - as has been shown - can be done by checking the validity of equation [2]. Since under thin-walled tube assumptions the stresses and hence the stress increments are linear functions of the load increments and the pressure increments, it is sufficient to check the validity of the form

$$\begin{aligned} d\epsilon_x &= a_x dL + b_x dp \\ d\epsilon_y &= a_y dL + b_y dp \end{aligned} \qquad [5]$$

where a's and b's are to be independent of dL and dp. Furthermore, as stated in the last section, the quantities actually recorded during a test run were the readings of dial gages. From these readings the numbers of least increments of each dial gage measured from some reference reading were computed for every entry. The nominal strains were computed from the number of least increments by multiplying them by an appropriate constant ($\varkappa$ and $-\lambda$). Again a multiplicative constant does not affect the validity of equation [5]; hence

$$de_x^p = de_x^p = \varkappa a_x dL + \varkappa b_x dp \qquad [6\,a]$$

$$de_y^p = -de_y^p = \lambda a_y dl + 2\lambda b_y dp \qquad [6\,b]$$

where e_x^p and e_y^p are the number of least increments from the reference readings in the x and y directions respectively.

Although [6] is the form that is desired, it is not the form obtained directly by the use of the computed increments of the dial readings. These latter contain also the contributions of the elastic strains. In order to obtain a set of increments that can be used in equation [6], it is necessary to subtract from the total increments recorded during a test those increments that represent elastic strains. To this end before and after each test run purely elastic check runs were made both with pressure variations only and with load variations only in order to determine the slope of the de^e vs. dL and the de^e vs. dp curves and the changes of these slopes during the test run. These data indicated that with no appreciable error the elastic coefficients could be considered constant. The elastic increments were computed from the load and pressure data and subtracted from the total increments. The differences were used as representing the plastic increments in equation [6].

Since elastic strains are linear functions of the load and pressure, it was not absolutely necessary to subtract them in order to check the validity of a linear form such as [6] . However, since a plastic stress-strain relation was sought, the results were more easily interpreted when free from the effects of the elastic behavior.

Equation [3] suggests the method of approach used in this analysis. If a loading function exists, the validity of linearity implies that at least locally it should be possible to make the plots e_x^p vs. L and e_y^p vs. L coincide by changing the e^p scale of one plot and by translating the plot vertically. The plots should coincide in spite of the fact that e_x^p and e_y^p are functions of L and p but are plotted against only L. If, therefore, the plastic strain scale of, say, e_x^p were reduced by a suitable factor q, the plots $q(e_x^p - e_{xo}^p)$ vs. L and $(e_y^p - e_{yo}^p)$ vs. L - where e_{xo}^p and e_{yo}^p correspond to some fixed values L_o and p_o - should coincide. Any deviation from coincidence would be indicative of non-linearity.

Comparison of the two plots was simplified by the introduction of $\Delta\theta = q\Delta e_x^p - \Delta e_y^p$ where $\Delta e^p = e^p - e^p_o$ indicates the number of least increments of the dial gage from the reference e^p_o. q was determined so as to make the overall plastic strains in the x and y directions équal for the entire test run:

i.e.

$$q = \frac{\text{overall } \Delta e_y^p}{\text{overall } \Delta e_y^p}$$

$\Delta\theta$ is therefore a measure of the deviation from coincidence of the two plots. Its units are the same as those of e_y. If linearity were valid, then $\Delta\theta$ would be zero - at least locally. If there were a systematic variation of $\Delta\theta$ with dL or dp, then linearity would be impossible.

Fig. 3 shows that for tube GH $\Delta\theta$ is essentially zero over its entire length. Linearity is possible, therefore, for this tube provided the magnitudes of the strains prove to be linear functions of the stress increments. Fig. 4 on the other hand shows for tube AB not only that $\Delta\theta$ is not zero, but that it is strongly dependent on the (incremental) pressure deviations. This dependence is evidenced by the simultaneous appearance of the loops in Fig. 4b and 4c. Clearly for this tube linearity is impossible, and the plastic stress-strain relation must be non-linear.

There seems to be little question of the existence of a loading function when time and temperature effects are absent. However, it is not necessary to use its existence to prove non-linearity in the case of tube AB. If linearity were valid, then $\Delta\theta$ vs. L (Fig. 4c) would have the same shape as the p vs. L curve with the exception of a local scale factor that - like the coefficients in equation [5] - could be a function of the load, pressure, and load-pressure history, but could not be a function of the load or pressure increments. Fig. 4b and 4c show that this is not the case. The general shape of the loops in 3c do not correspond with those in 3b. The peak points in the two sets of loops do not correspond. Although there can be no doubt that the loops of 3c are closely related to those in 3b, there is slight but definite lag in both the beginnings and ends of those in 3c relative to those in 3b. It would not be possible to find coefficients in equation[2]which are functions of the load and pressure only that would allow a linear transformation of 3b and 3c within the limits of experimental error. Thus with or without the assumption of a loading function, the plastic stress-strain relation for tube AB must be considered non-linear.

Strictly speaking, it is not necessary that there be no dependence of $\Delta\theta$ on what has here been considered increments of pressure, dp, even if a smooth loading function is assumed to exist. This is true since in reality the measured "dp's" are finite increments of p. In this eventuality, however, the variations of $\Delta\theta$ with p would have been of a smaller magnitude. Variations of this magnitude could occur of course for loading functions whose surfaces have a pointed vertex, but non-linearity in this case is assured as seen previously. Finally a similar argument could be applied to dependence of $\Delta\theta$ on L or dL.

To establish the validity of linearity for tube GH it would now be sufficient to examine the dependence of only the magnitudes of the plastic strain increments on the stress increments. However, the computations for the direct determination of the coefficients in equation[2]were just as simple and were, therefore, made.

The method consisted in first approximating the overall curve in Fig. 2a by a smooth curve. The deviations from this curve were then matched by a constant times the pressure for each cycle. The slope of the smooth curve (together with q, x, and λ) gave the a's equation [9], and the multiplier of the pressure gave the b's. The b's were not constant throughout the test, to be sure, but could be considered constant within any one cycle to the degree of accuracy looked for. The fit obtained by this process was within the experimental error except in the neighborhood of points of unloading. In these regions the lack of correspondence was felt to be due more to the hysteresis in the testing machine than to a non-linearity in the magnitudes of the strain increments. (In reference to hysteresis in the testing machine, see reference (6).) It was felt that linearity was well justified in the case of tube GH.

Smooth and Pointed Loading Surfaces for Tubes GH and AB

The question is still open as to what type of loading surface best fits the data presented. To investigate this question consider the differential of $\Delta\theta$: $d\theta = q de_x^p - de_y^p$. Recalling the definition of $\varkappa$ and λ :

$$q de_x^p - de_y^p = q\varkappa d\varepsilon_x + d\varepsilon_y^p .$$

Define $k_\alpha = (q\varkappa, \lambda)$. Then $q de_x^p - de_y^p$ may be considered as the scalar product of $d\varepsilon^p$ and k_α; that is

$$\begin{aligned} q de_x^p - de_y^p &= d\varepsilon_\alpha^p k_\alpha \\ &= |d\varepsilon_\alpha^p|\,|k_\alpha|\cos\zeta \end{aligned}$$

where ζ is considered the angle from k_α to $d\varepsilon_\alpha$. The cross product of k_α and $d\varepsilon_\alpha^p$ is

$$\begin{aligned} \epsilon_{\alpha\beta} k_\alpha d\varepsilon_\beta^p &= -\lambda d\varepsilon_\alpha^p + q\varkappa d\varepsilon_y^p \\ &= |d\varepsilon_\alpha^p|\,|k_\alpha|\sin\zeta \\ &= \frac{\lambda}{\varkappa} de_x^p - \frac{q\varkappa}{\lambda} de_y^p \end{aligned}$$

where $\epsilon_{\alpha\beta}$ is the two dimensional alternating tensor. Dividing

$[qde^p_x - de^p_y]$ by$[-\frac{\lambda}{\varkappa q}(qde^p_x) - \frac{q\varkappa}{\lambda} de^p_y]$ gives an expression for the $\cot\xi$. Note that k_α makes an angle of 90° with the direction of the overall $\Delta\varepsilon^p_\alpha$. If χ is defined to be the angle from the overall $\Delta\varepsilon^p_\alpha$ to $d\varepsilon^p_\alpha$, then $\chi = \xi - 90^\circ$. Hence

$$\frac{\lambda}{\varkappa q} \tan\chi = \frac{qde^p_x - de^p_y}{qde^p_x + (\frac{\varkappa q}{\lambda})^2 de_y} .$$

Although it is not feasible to make a direct computation of $\frac{\lambda}{\varkappa q}\tan\chi$ for every set of increments, the slope of the plot $(q\Delta e^p_x - \Delta e^p_y)$ vs. $(q\Delta e^p_x + (\frac{\varkappa q}{\lambda})^2 \Delta e^p_y)$ [i.e.$\Delta\theta$ vs. $\Delta\psi$] gives the $\tan\chi$ to within a constant scale factor. Fig. (5a) shows this plot for tube AB. A similar plot for tube GH is not shown since the variation in $\Delta\theta$ (Fig. 3c) is too small. Since $(q\Delta e^p_x + (\frac{\varkappa q}{\lambda})^2\Delta e^p_y)$ $[=\Delta\psi]$ is not a constant times the load, a plot of p vs. $\Delta\psi$ was also given for ease in correlation of the first plot with the previous data (Fig. 5b).

There are many interesting observations concerning (Fig. 4 & 5). The first is that the magnification of very small strain differences represented in the $q\Delta e^p_y - \Delta e^p_y$ direction has not masked the consistency of the behavior.

The other points of interest concern the shape of the bumps. Predominantly they are composed of three principal slopes: an initial slope up, a horizontal slope at the top (missing in some cases) followed by a slope down. The slope down in the last two bumps is made up of two slopes, the steeper coming first. The initial slope has no relation to the slope of the corresponding pressure slope. The final slope seems to break into two slopes or to oscillate between two slopes if the corresponding pressure slope is too flat. With the exception of the transition from the downward slopes to the horizontal slopes, the transition points are well defined, i. e. the change in slopes is definite. The bumps themselves become flatter as the run proceeds: the angles between the initial and the final slopes become greater. The progress of the change is itself interesting. The first two bumps are almost scale models of each other. The next two have roughly the same initial slope but much more gentle final slopes. The last bump has gentle initial and final slope. Finally, the points of slope change do not seem to coincide exactly with the points of direction change of the loading path, but seem to lag a short distance behind them. At the top of the bumps there seems to be no correlation between the loading direction and the change to or from the horizontal portion of the bumps at all.

If the loading function associated with the tube in question were smooth, and if in reality the increments of pressure cannot be assumed to be infinitesimal, then an increase in pressure from the load-pressure state used in the run here would result in a decrease in the angle that the normal to the loading surface makes with the horizontal, i.e., the normal would be rotated in a counterclockwise sense. However, these angle changes would be of smaller order of magnitude than those observed.

To illustrate the point, Table I was prepared. For each loading cycle the ranges of the ratio $d\epsilon_v^p/d\epsilon_x^p$ and of the corresponding angle, η, that the strain increment vector makes with the horizontal is entered on the one hand, and on the other the same quantities computed on the basis of a J_2 theory. The angles in the former entry were computed from the measured slopes of the $q\Delta e^p_x - \kappa e^p_y$ vs. $q\Delta e^p_x + (\frac{\kappa q}{\lambda})^2 \Delta e^p_y$ plot. The angles in the prediction of the J_2 theory seem a few degrees large; but such a constant error could easily be accounted for in the frame work of a linear flow theory: e.g. by the inclusion of a J_3 term. The magnitudes of the changes, on the other hand, are of the correct order for any smooth curve to which J_2 is a reasonable approximation. Clearly this magnitude is not the required one. The largest angular difference in the entire J_2 part of the table is 1.4°, while the smallest observed angular difference for any single cycle was 9.9°.

Another demonstration illustrating the same conclusion is provided in Fig. 6. Here an idealized loading cycle, p vs. L, is given in Fig. 6a. In Fig. 6b the pointed curve represents the corresponding prediction of the $q\Delta e^p_x - \Delta e^p_y$ vs. L plot for a cornered loading surface. It is assumed for the sake of simplicity of calculations that for $dp = 0$ the loading direction of the strain increment vector $d\epsilon^p_\alpha$ makes an angle of -22.8° with the horizontal. When $dp>0$, the direction of $d\epsilon^p_\alpha$ (as well as the normal to the loading surface) makes an angle of -5.4° with the horizontal. Finally, when $dp<0$, the angle for both the straining direction and the normal is -29.3°.

The flatter curve in Fig. 6b shows the corresponding prediction of a loading surface that has the same changes of direction of the normal to the loading surface as a J_2 theory would for the pressures and loads of this cycle. The surface has been tilted relative to the J_2 theory surface, however, in order that $q d\epsilon_x^p - d\epsilon_y^p = 0$ when $dp = 0$.

The loading cycle in Fig. 6a was chosen as a crude but simple approximation to the second loading cycle in Fig. 4b. Examination of the pointed curve in Fig. 6b shows that the general features of both shape and size of even this approximation are in fair agreement with the actual case. The flatter curve, on the other hand compares poorly in both features, but in particular with respect to the magnitudes of the ordinates. It should be noted in passing that after several cycles such as the one presented here the difference in the two predictions would become obscured by the cumulative effects and would not stand in such sharp contrast.

The arguments above lead to the conclusion that a corner does exist on the loading surface of the tube AB. Analogous arguments for the tube GH lead to the conclusion that its loading surface does not have a corner large enough to be detected by the analysis presented here. It is therefore considered smooth. (The former result again precludes the possibility of linearity in the case of the tube AB.)

Discussion

The method used in the investigation of the corner in the loading surface was not the only one available. Another way, for example, was to plot ϵ_y^p vs. ϵ_x^p and then to determine whether or not there were any local variations that could be correlated with changes in the direction of loading. If the local variations were absent, then there would be no corner in the loading surface. This method was followed in reference (5). The plots were good approximations to straight lines in that the deviations from straight line approximations fitted to the points were small. It was felt at the time that the smallness warranted the conclusion that the loading direction was constant. The more sensitive analysis presented here shows this conslusion not tenable in the case of tube AB, although, of course, it is for tube GH. At best the plot of ϵ_y^p vs. ϵ_x^p is insensitive compared to the plot $\varphi\Delta\sigma_x^p - \Delta e_y^p$ vs. $\varphi\Delta e_x^p + (\frac{xg}{\lambda})^2 \Delta e_y^p$.

Insofar as the author is aware the literature gives no report to date of the observance of corners on loading surfaces for polycrystalline materials. There apparently are two reasons why this may be so. First and most important, the corners were usually not looked for. The second reason stems directly from the first, most experiments have not been so designed that the corners would show. The usual experiment that concerned itself with loading surfaces has been designed to show for a given loading direction at what stage yield took place. After yield had been reached, a loading path of arbitrarily changing direction has not been followed. Yet this type of path is the only one that can show a corner completely. Some tests have followed a loading path of varying direction, but one that turns in one direction only. At best such a path could pick up the effect of only one side of a corner.

It is interesting to note that in the case of tube AB after the corner was formed (assuming that the original loading surface was smooth) the effect of loading in many different directions from the corner had the effect of flattening it. If the original surface was smooth, the loading in one direction seemed to have a tendency to form a pointed vertex, while loading in many directions from the vertex seemed to destroy it.

It should be emphasized that the corner did not appear in both tubes. It cannot be inferred that for a given material there will be or will not be a cornered vertex in the loading surface.

With regard to the direction of the strain increment vector associated with a corner, no simple rule for its determination was obvious.

Creep

In reference (5) it was assumed that the plastic creep and strain increment vector were closely parallel, and that the creep effects were not important in the overall trends. It is interesting after closer study to consider the problem of creep again.

Creep as usually defined is that permanent deformation of a body that occurs under and due to constant loads. It is thought of as occurring in three phases: primary or transient creep, secondary or steady creep, and tertiary or accelerating creep. Primary creep exerts its influence immediately after the cessation of loading and is characterised by a decreasing strain rate. Secondary creep has an essentially constant strain rate that is the minimum of the three types discussed. Tertiary creep begins with the first increase of strain rate after secondary creep. This discussion is concerned only with the first mentioned type since, as mentioned in (5), tests indicated that secondary creep effects were negligible, and since the test was not conducted over a long enough period for tertiary creep effects to occur.

If a given material is loaded into the plastic region, and if the loading is then stopped, but held constant, creep will usually occur. Questions present themselves as to what happens if the loading instead of being made zero were to have its rate decreased to a small figure, or were changed in direction. Although these questions as yet are not answered, it seems as reasonable to the author that an immediately preceding loading should add an additional permanent strain to the strains arising from a state of subsequent loading as that it should add an additional permanent strain to the zero permanent strain arising from a subsequent cessation of loading.

From an experimental point of view, however, there is a vast difference. The time dependent permanent strains in the latter case can easily be differentiated from zero strains, while in the former case there is as yet no decisive way to distinguish which permanent strains are time dependent on the preceding loading and which are caused by the new state of loading.

It might be expected that after deviation from one direction of loading that has been followed for a while the time dependent straining resulting from that loading would appear as a tendency for the material to continue straining in the same way. Put another way it might be expected that there would be a time lag between the change of loading direction and the change of straining direction associated with it. The loops in Fig. 3c give evidence of a small lag between these changes. The lags are much too small to invalidate the conclusions drawn before; and as stated in (5) the overall effects were not influenced by creep.

Conclusion

Careful examination of two thin-walled tubes shows that the assumption of linearity in the plastic stress-strain law is justified within the experimental accuracy for one tube, while it was not justified for the other. The former tube possessed a smooth loading surface while the latter had a definite corner. Techniques now exist for the investigation of loading surfaces for corners.

Time effects appear in the analysis. Even though the loading was not stopped insofar as was possible, effects in many ways analagous to creep came into evidence. These effects were not of sufficient magnitude to invalidate the conclusions stated in the previous paragraph. Clarification of the effect analagous to creep for continuous but varying loading paths requires additional study.

TABLE I

Cycle	Corner from	Corner to	Corner $d\varepsilon^p_y/d\varepsilon^p_x$ from	Corner $d\varepsilon^p_y/d\varepsilon^p_x$ to	J_2 from	J_2 to	J_2 $d\varepsilon^p_y/d\varepsilon^p_x$ from	J_2 $d\varepsilon^p_y/d\varepsilon^p_x$ to
1	-31.4	-14.3	-.610	-.255	-25.8	-25.1	-0.485	-0.468
2	-27.1	- 5.5	-.512	-.096	-25.9	-24.4	-0.485	-0.454
3	-25.1	-15.2	-.469	-.272	-25.9	-24.8	-0.485	-0.463
4	-28.1	-15.6	-.535	-.280	-25.9	-25.3	-0.485	-0.472
5	-27.5	-17.2	-.521	-.310	-25.9	-24.8	-0.485	-0.460

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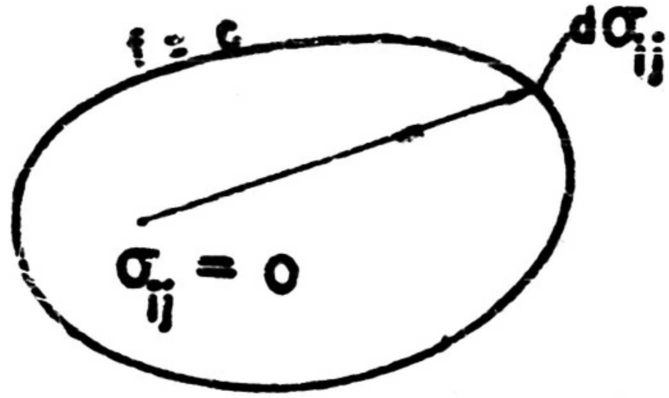


FIG. 1

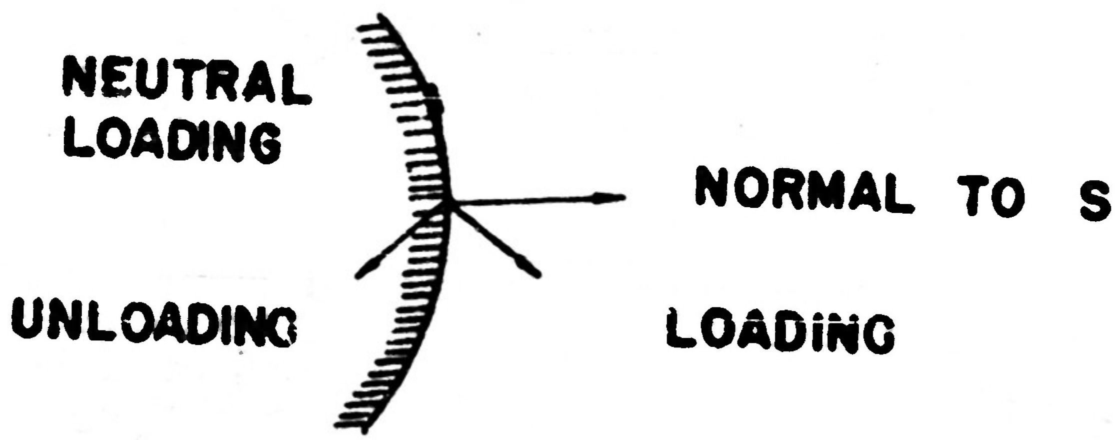


FIG. 2a

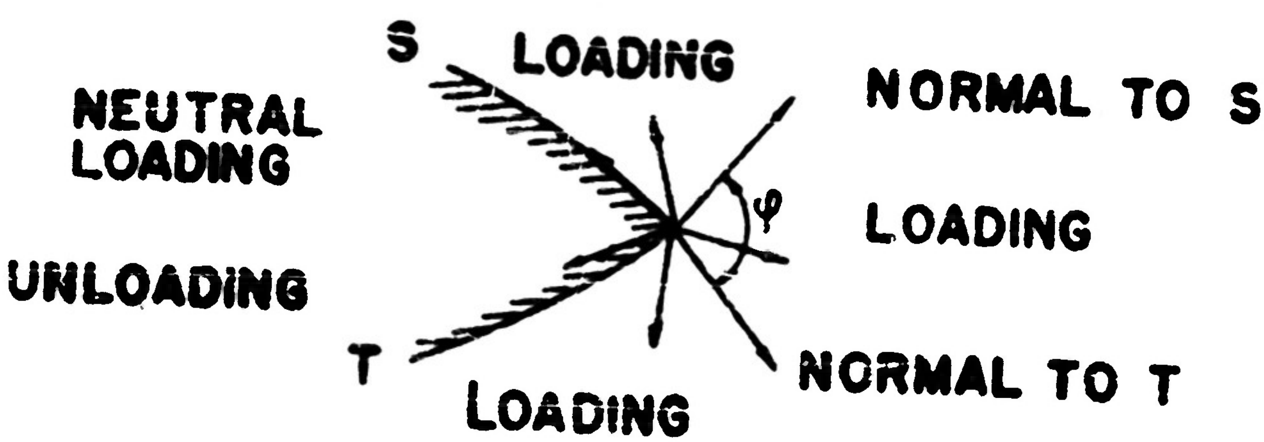


FIG. 2b

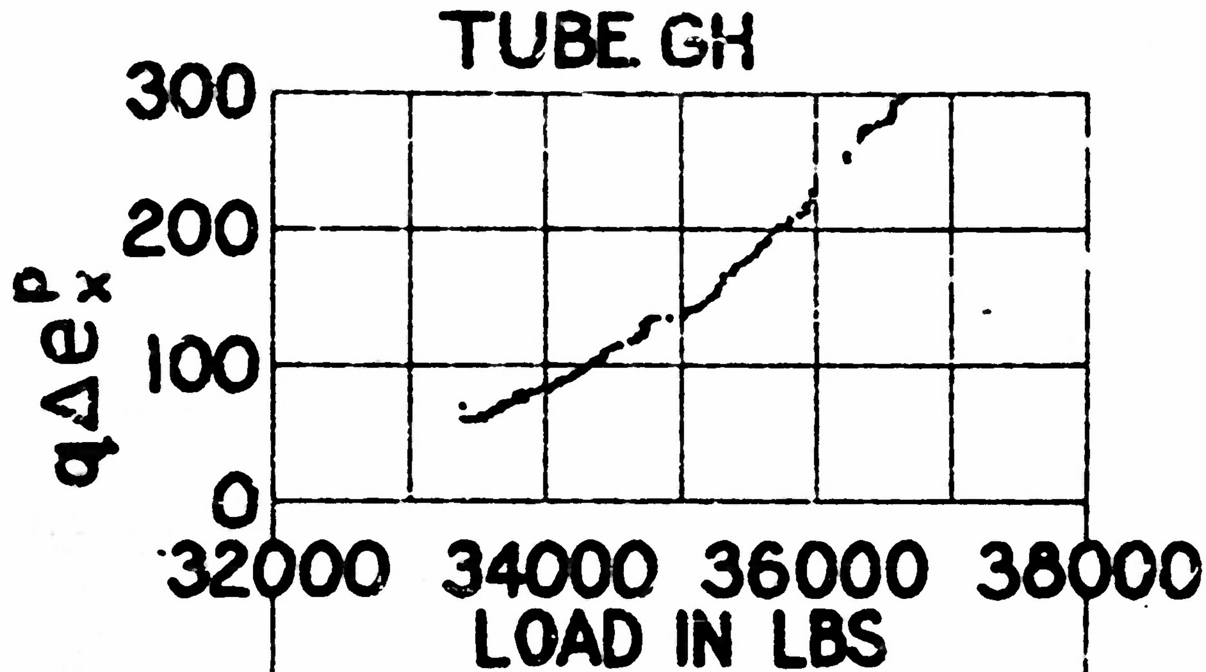


FIG. 3a

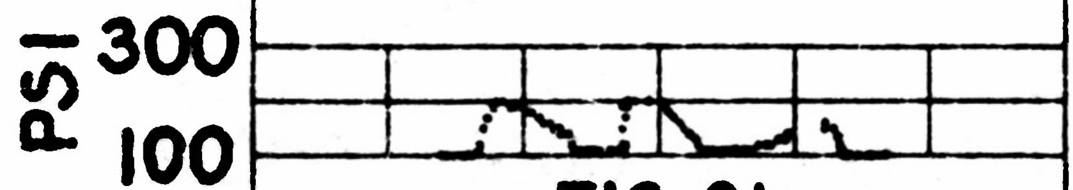


FIG. 3b

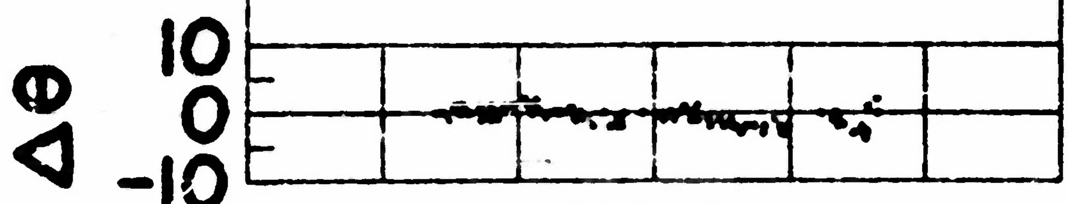


FIG. 3c

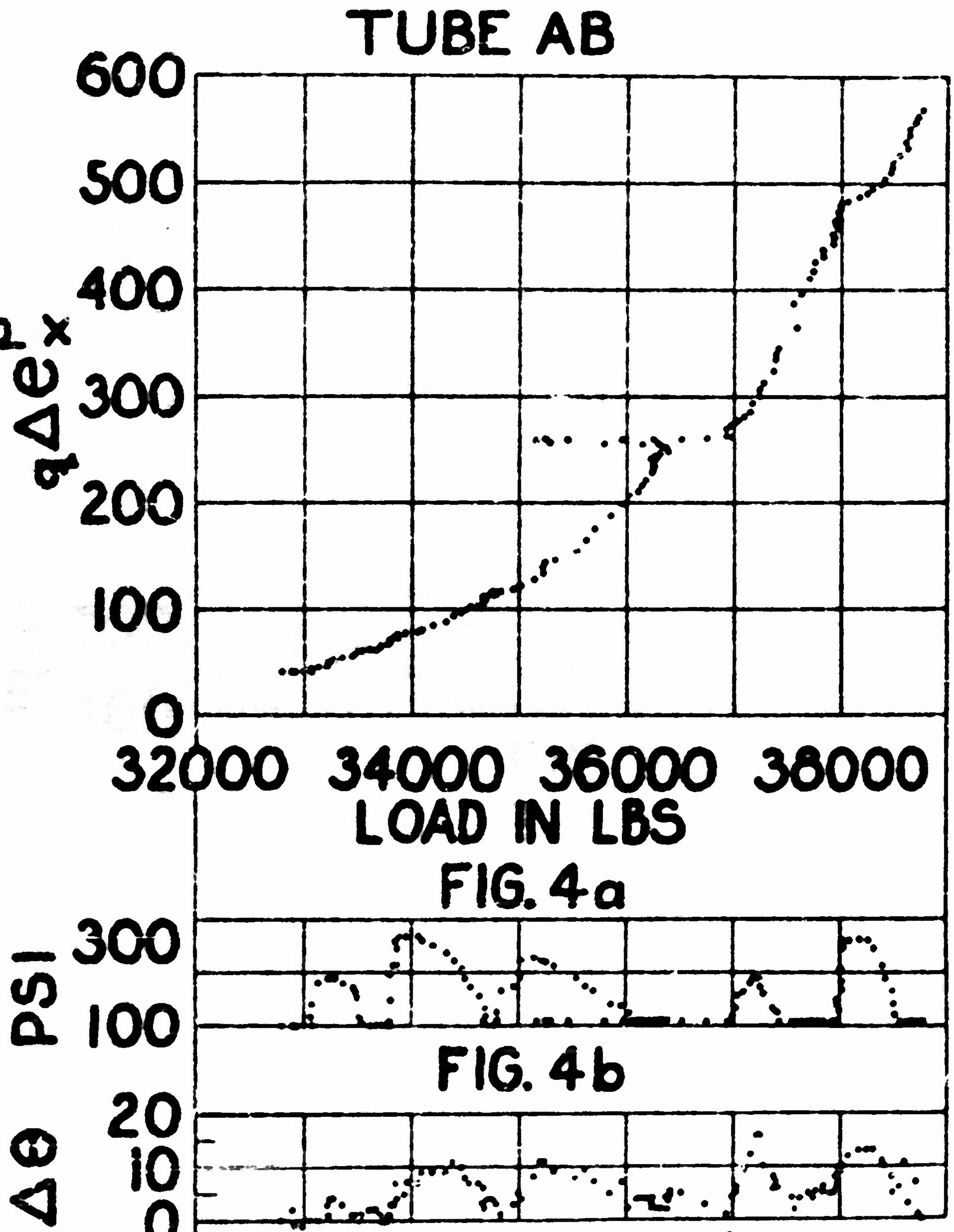


FIG. 4a

FIG. 4b

FIG. 4c

TUBE AB

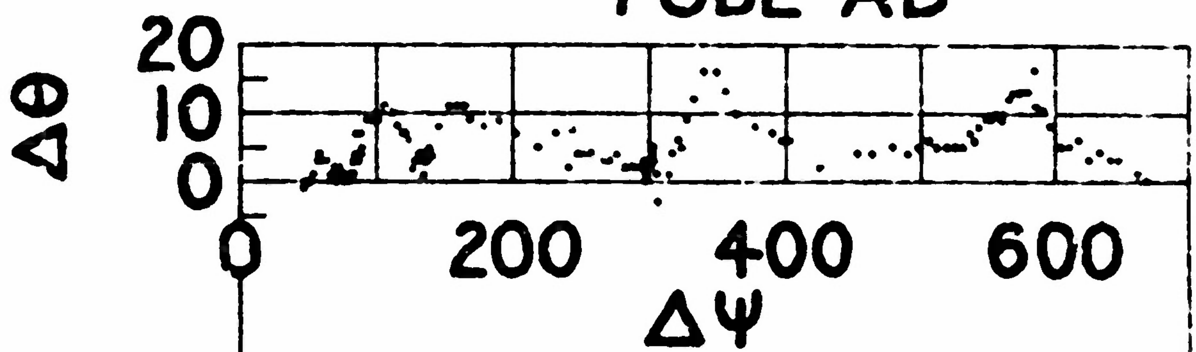


FIG. 5a

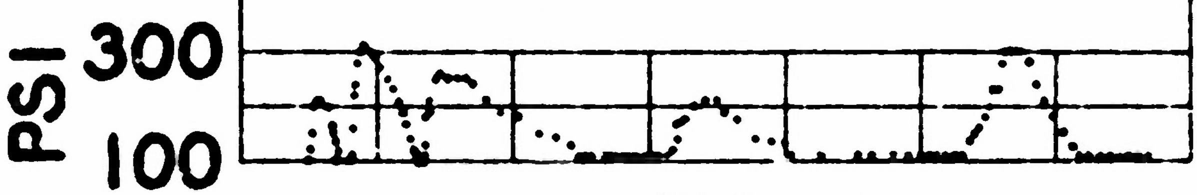
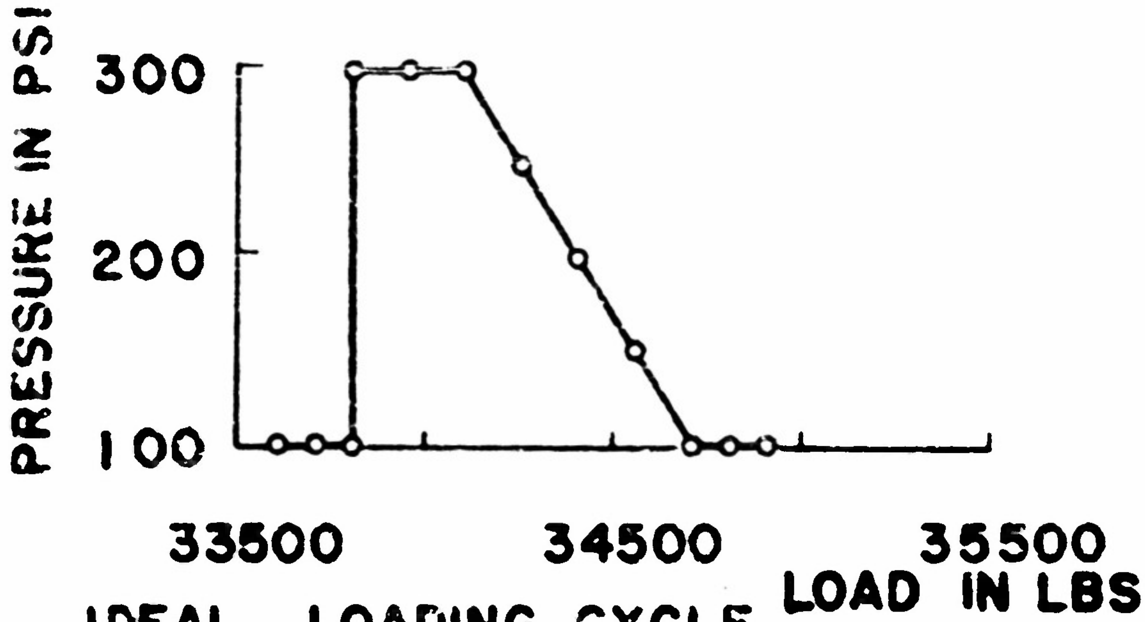


FIG. 5b



IDEAL LOADING CYCLE

FIG. 6a

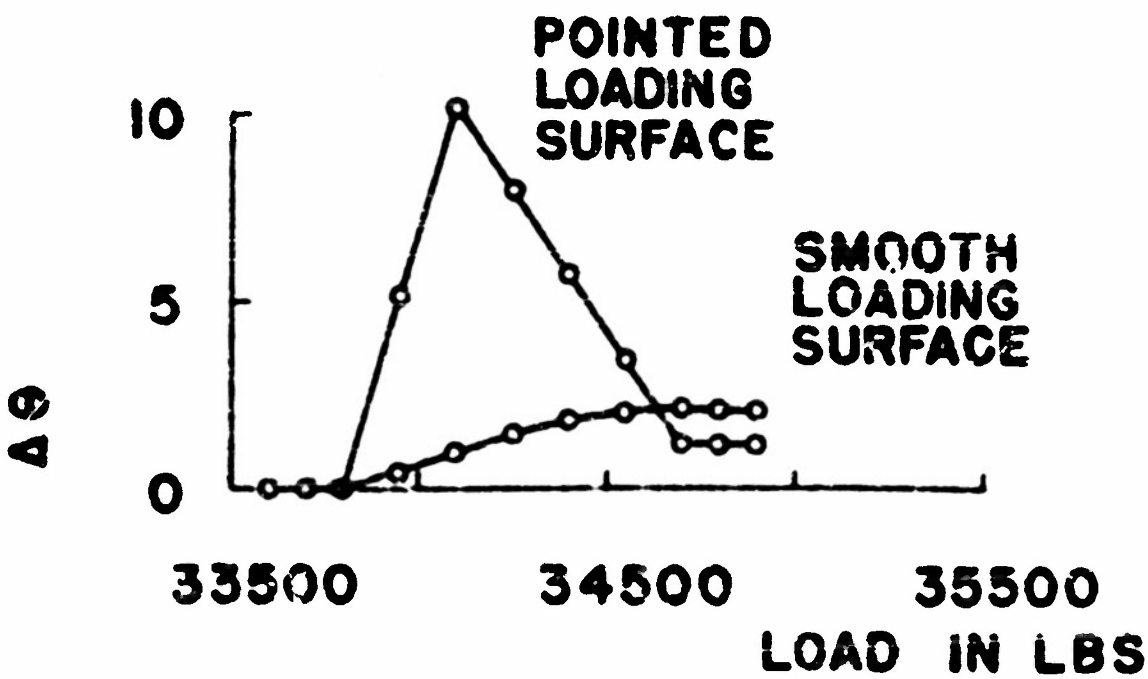


FIG. 6b

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